Department of Physics and Astronomy, University of Canterbury,
Private Bag 4800, Christchurch, New Zealand

Cosmological Background and Perturbations

Hayden Lee

UC
UNIVERSITY OF CANTERBURY
Te Whare Wānanga o Waitaha
CHRISTCHURCH NEW ZEALAND

MAPH480 Honours project 2011

Supervisor: Assoc. Prof. David L. Wiltshire and Dr Teppo Mattsson

Abstract

We review the linear cosmological perturbation theory and the recent work of Bičák, Katz and Lynden-Bell on developing the Machian gauge. This gauge incorporates Mach's principle in the sense that the acceleration and rotation of local inertial frames are determined by the energy-momentum perturbations. We also discuss the possibility of the existence of other Machian gauges.

One application is to examine whether it accounts for structure formation by considering scalar perturbations. We study the behaviour of density perturbations in two models: matter- and radiation-dominated universe.

Contents

Chapter 1

Introduction

Cosmology is one of the oldest interests of mankind. How did the universe begin? Many philosophers and physicists have been attempted to give an answer to this question since the ancient Greeks. Since the formulation of general relativity by Einstein in 1915, cosmology has developed rapidly and new ideas are still actively being developed. And it would be no exaggeration to say that cosmology is having its golden age at the present. As technology is advancing day by day, our measurements are becoming more and more precise. The past quarter of a century was remarkable in the sense that many productive observatories have been launched including the Hubble space telescope, Chandra X-ray observatory, Sloan Digital Sky Survey (SDSS), and Wilkinson Microwave Anisotropy Probe (WMAP). During this time, there was a ground-breaking measurement of anisotropies in the cosmic microwave background (CMB) which has provided a whole new framework for viewing the universe. This almost isotropic thermal radiation – one of the remnants of the big bang – has given a significant evidence for establishing the ΛCDM model as the current standard model of big bang cosmology. The values of the parameters giving the universe (such as the Hubble parameter describing the expansion rate) are becoming more and more tightly constrained, and people say that we have reached the "precision era" of cosmology. However, we need to question whether this strengthens our understanding of the universe or not, i.e., whether we are reaching towards the "accurate era" of cosmology. There are still open challenges remaining to the standard model, and cosmologists are thus looking for other viable models which might resolve various puzzles as well as being consistent with our current observations. Nevertheless, no theories are yet supported sufficient observational evidences to be superior to the standard model. Therefore, whether we are arriving at a precision or accurate era of cosmology remains a big issue.

The main assumption of the standard model is that the universe is well described by a spatially homogeneous and isotropic geometry. However, the matter distribution of the present universe is nowhere near to being homogeneous on local scales. (For a trivial example, consider our solar system; we have the sun in one direction, and the moon in another.) We can only approximate the universe as being statistically homogeneous at very large scales, typically of order bigger than $110h^{-1}$ Mpc where h is a dimensionless Hubble parameter, $H_0 = 100h^{-1}$ km s^{-1} Mpc^{-1}. It is important to understand how this inhomogeneity first came about and how structures such as stars and galaxies are formed as the universe evolved. These questions may be answered by cosmological perturbation theory.

A brief history of cosmological perturbations

The remarkable measurement of the CMB by WMAP in the last decade suggests that the universe was very close to being perfectly homogeneous and isotropic at the epoch of last scattering. Hence, it is important study how the inhomogeneities we observe today have been growing since the epoch of last scattering. The following is a brief outline of the procedure:

1. **Initial conditions**. We used to have no precise understanding of how primordial fluctuations were generated; they were rather proposed to fit the observational data. Today we have a good candidate scenario for the generation of such fluctuations: inflation. The universe has undergone an extreme expansion a short time after the big bang, driven by a hypothesized inflaton field. Although we still do not have precise understanding of inflation, it is assumed that primordial fluctuations were generated by vacuum quantum fluctuations in this inflaton field.

2. **Recombination and last scattering**. As the universe expanded, there was a point when the temperature of the universe had sufficiently cooled down that free electrons and protons formed hydrogen atoms. This represented a phase transition: photons which previously scattered from free electrons were now free to propagate arbitrarily long distances without scattering; these photons are now observed as the CMB. Measurement of the CMB today shows that the last scattering occured at a redshift $z \approx 1100$ when the universe about 380,000 years old. The CMB has a perfect blackbody spectrum with a mean temperature which is uniform to 1 part in 1000. If we subtract off a dipole – believed to be due to our peculiar motion with respect to the cosmic rest frame – then the remaining temperature fluctuations are extremely small, at the level of 1 part in 100,000. These temperature fluctuations are believed to be due to variations in redshift created by the gravitational potentials of density fluctuations in the baryon-photon plasma of order $\delta\rho/\rho \sim 10^{-5}$.

3. **Structure formation**. The small fluctuations in the primordial fluid grew as the universe evolved. Because of the attracting nature of gravity, a small overdense regions attracted the surrounding matter. When the self-gravity of an interstellar gas cloud exceeds the internal gas pressure, i.e., when the gravity counteracts strongly enough to counteract the thermal energy that causes the gas to expand, the cloud collapses allowing objects such as stars to form. This so-called gravitational instability allowed subsequent generations of star formation and galaxy formation to occur; therefore, forming the structures of the universe we observe today.

The above facts motivate us to consider perturbations of the Friedmann-Lemaître-Robertson-Walker (FLRW) model of the universe at the epoch of last scattering. Since the pioneering work of Lifshitz [1] in 1946, the subject has been developed by various workers including an influential paper of Bardeen [2] in 1980. The work of Lifshitz included considering a small deviation of the metric tensor from that of the FLRW model. However, there are some subtleties. First, the metric tensor is not directly measurable, and second, it contains degrees of freedom which correspond to a mere coordinate transformation of the background. Hawking [3] attempted to resolve this by considering perturbations of actual physical quantities such as curvature and density. However, as pointed out by Bardeen, he failed to recognize that constant time slices cannot be orthogonal to worldlines of the fluid-element when pressure and density perturbations are present. Instead, to understand the possible subtleties raised, we need to fix a specific coordinate system and adopt suitable conditions on the metric tensor. We call this ‘fixing a gauge’.

There are many gauge choices we can have, and each of these has its own physical interpretation. One popular choice is the synchronous gauge, first considered by Lifshitz, which has the property that there exists a class of comoving observers who freely fall at a fixed spatial coordinate. This assumption is well-justified in the early universe before structure formation. However, it leads to problems when geodesics of such observers intersect each other – which happens to be true when stars form through gravitational instability – in which case we have a singularity. Recently, a possible class of gauge choices was introduced by Bičák, Katz and Lynden-Bell [4] with the intent of incorporating Mach’s principle into the linear cosmological perturbation regime. At this point, let us consider two questions: What is Mach’s principle? How does Mach’s principle relate to current cosmology?

What is Mach’s principle?

From Newton’s perspective, space was an absolute entity. Although questionable, this concept prevailed until a serious challenge came from Ernst Mach, a physicist and philosopher, almost two hundred years later. Broadly stated, Mach’s idea was, “space and matter are relational things.” The idea is well-illustrated by a simple thought experiment: imagine yourself spinning alone in the universe; you see distant stars whirling around you as you spin and you feel that your arms get pulled outward. Would you feel it when there are no stars at all, i.e., when you are in a completely empty space? This somewhat philosophical idea gave a foundational inspiration to Einstein to formulate his theory of relativity. In his paper, Einstein stated it as “the entire inertia of a point mass is the effect of the presence of all other masses, deriving from a kind of interaction with the latter [5].” He tried to fully embody this principle to his theory of relativity but did not quite succeed – the closest attempt being the strong equivalance principle which states, “at any event, it is possible to choose a local inertial frame such that in a sufficiently small spacetime neighbourhood all non-gravitational laws of nature take on their familiar forms appropriate to the absence of gravity, namely the laws of special relativity.”

There is still skepticism towards Mach’s principle, with a famous counter-example of the Gödel metric: a solution of field equations which describes a homogeneous universe with uniform matter distribution. In this universe, there exists a preferred axis of rotation, whereas rotation should only depend on matter distribution – hence, violating Mach’s principle. Furthermore, Rindler pointed out that the Lense-Thirring effect – a relativistic correction to the precession of gyroscope near a large rotating mass [6], [7] – exhibits an anti-Machian nature [8]. However, triggered by the conference *‘Mach’s principle: from Newton’s bucket to quantum gravity’* in 1993 in Tübingen, the first conference solely devoted on Mach’s principle, various interpretations of Mach’s principle have been introduced and discussed. Moreover, in their paper, Bondi and Samuel [9] stated ten distinct interpretations where they stress that Rindler’s interpretation is just one amongst many. In particular, the version of Mach’s principle introduced by Bondi [10] in 1961 which says, “local inertial frames are determined through the distributions of energy and momentum of the universe by some weighted averages of the apparent motions,” does not contradict the Lense-Thirring effect.

Mach’s principle is a broad idea that cannot be tested directly, and the question of whether it is valid remains open. The very nature of space and time has always been an interesting question from both physical and philosophical perspective. After Newton’s absolute space was abandoned, we now know that spacetime and matter are interrelated, the bizarre nature that we were never able to think about before Einstein. Thus, it may well be the case that Mach was right. In this perspective, we should not disregard Mach’s principle, and we may embody this principle whenever appropriate. In particular, we would like to know applicability of Mach’s principle to current cosmology.

How does Mach’s principle relate to current cosmology?

As stated earlier, the standard model of cosmology exhibits many subtleties. In particular, we might question the main assumptions of the ΛCDM model: the global spatial homogeneity and isotropy of the universe. A major goal in inhomogeneous cosmology is to obtain a viable model of the universe, which agrees with the observational data, without these assumptions. However, the primary difficulty we face is that the pure (Einstein) field equations are very difficult to solve because of their nonlinear nature. Without special symmetry assumptions, we face insurmountable mathematical difficulty in solving the equations. For this reason, we need the notion of averaging in inhomogeneous cosmology.

Ellis [11] was the first to emphasize the importance of the “fitting problem,” namely that even if there is some notion of homogeneity on large scales in a statistical sense, then this may still differ from the geometry of an exactly homogeneous and isotropic universe, and furthermore the problem of fitting the local geometry into the average geometry may be highly nontrivial. Building on other work, Buchert [12], [13] formulated a particularly simple averaging scheme which deals with volume averages of scalar quantities. Now, the question we have raised was: how can we relate Mach’s principle to current cosmology? The phrase ‘some weighted averages’ in Mach’s principle was never clearly understood. However, as argued by Wiltshire [14], this may naturally relate to averaging in inhomogeneous cosmology, and to confirm the connection, one might incorporate Mach’s principle in the context of averaging an inhomogeneous cosmology and see if it agrees with the observational data. However, since the Buchert average is only defined for the synchronous gauge which does not exhibit a Machian nature, a new averaging formalism needs to be constructed based on a gauge condition which does exhibit Machian nature.

Machian gauges, proposed by Bičák, Katz and Lynden-Bell, incorporate Mach’s principle in the sense that the acceleration and rotation of local inertial frames are determined by the perturbed energy-momentum tensor. This project reviews linear cosmological perturbation theory and the work of Bičák, Katz and Lynden-Bell. The formulation of the Machian gauge and the solutions of the field equations are examined. My explicit contributions are made in §4.2, where their work is applied to scalar perturbations to examine structure formation. My understanding of Machian gauges and the possibilities of the existence of other Machian gauges is discussed in §3.4. Additional comments are given whenever possible.

Chapter 2

Cosmological Perturbation Theory

In this chapter we will review the basic concepts of linear cosmological perturbation theory. For a more complete review, see, e.g., [15]. For a gauge-invariant approach, see [2] and [16].

2.1 Metric perturbation

We consider the 3+1-decomposition of the 4-dimensional spacetime manifold, $\mathcal{M}$, describing the universe, into a one parameter family of spatial hypersurfaces Σ_t, where the parameter is chosen to be the global time coordinate t. This is possible in general if $\mathcal{M}$ is *globally hyperbolic*. In this case, we can write the 4-dimensional metric in *Gaussian normal coordinates*:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + \gamma_{ij}dx^i dx^j, \tag{2.1}$$

where γ_{ij} is the intrinsic 3-metric on the spatial hypersurfaces Σ_t induced by the 4-dimensional metric, $g_{\mu\nu}$. With such a decomposition, we are able to write the FLRW metric, describing the geometry of the spatially homogeneous and isotropic universe[1], in Cartesian coordinates $x^\mu = (t, x, y, z)$ as[2]

$$ds^2 = {}^{(0)}g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \tag{2.2}$$

where $a(t)$ is a scale factor. However, we can only approximate the universe as being perfectly homogeneous and isotropic at the epoch of last scattering. Hence, the metric describing the exact geometry differs from the above, but we still assume that the average geometry is described by it. Thus, we take (2.2) as the fictitious background metric, and the metric of our real universe at epochs near last scattering is described by a linear perturbation in metric

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu = ({}^{(0)}g_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \\ &= -(1 - h_{00})dt^2 + h_{0i}dt dx^i + a^2\left[(\delta_{ij} + h_{ij})dx^i dx^j\right]. \end{aligned} \tag{2.3}$$

[1] We will assume flat geometry $k = 0$, which is indeed a good approximation in the early universe, and work in units where $c = 1$ throughout the project. We will also use the symbol k in §4 to represent the comoving wavelength but no confusion shall arise.

[2] Latin indices i, j, etc. run from 1 to 3, whereas Greek indices μ, ν, etc. run from 0 to 3. We use the convention that ${}^{(0)}$ denotes the background quantities.

It is often useful to work in terms of conformal time η rather than the cosmological time t. The metric (2.3) in the coordinates $\tilde{x}^\mu = (\eta, x, y, z)$ is

$$\begin{aligned} ds^2 &= \tilde{g}_{\mu\nu} d\tilde{x}^\mu d\tilde{x}^\nu = ({}^{(0)}\tilde{g}_{\mu\nu} + a^2 \tilde{h}_{\mu\nu}) d\tilde{x}^\mu d\tilde{x}^\nu \\ &= a^2 \left[(-1 + \tilde{h}_{00}) d\eta^2 + 2\tilde{h}_{i0} d\eta dx^i + (\delta_{ij} + \tilde{h}_{ij}) dx^i dx^j\right], \end{aligned} \tag{2.4}$$

and the transformation between vectors V^μ and $\tilde{V}^\mu$ in the two coordinate bases is given by

$$\tilde{V}^\mu = \Lambda^\mu{}_\nu V^\nu, \tag{2.5}$$

where

$$\Lambda^0{}_0 = a, \quad \Lambda^i{}_0 = \Lambda^0{}_i = 0, \quad \Lambda^i{}_j = \delta^i{}_j. \tag{2.6}$$

We can then transform second rank tensors accordingly:

$$\tilde{W}^\mu_\nu = \Lambda^\mu{}_\alpha \tilde{\Lambda}^\beta{}_\nu W^\alpha_\beta, \tag{2.7}$$

and so on for higher rank tensors, where $\tilde{\Lambda}$ denotes the inverse of Λ.

There is an important distinction between the usual FLRW model and the perturbed model. The FLRW model considers the universe to be perfectly homogeneous and isotropic which means the metric and, therefore, the equations and the quantities are exact. However, the perturbed metric (2.4) ignores the second and higher order terms, which implies that even though it is a good approximation near the epoch of last scattering, it is still not exact. Hence, the perturbed quantities in this framework are not exact. So our usual understanding involving such quantities may not hold in this framework, and certainly the results break down after a time such that a substantial amount of inhomogeneities has grown.

2.2 Gauge transformation

In cosmological perturbation theory, one considers two distinct manifolds: background and perturbed spacetime. To express the perturbations of physical quantities in the perturbed spacetime, we must know how to relate the coordinates between these two manifolds. We can do this by defining a diffeomorphism between these two manifolds[3].

Let y^μ be the coordinates on the background spacetime $\mathcal{M}$. Then a diffeomorphism $\mathcal{D} : \mathcal{M} \to \mathcal{N}$ induces the coordinates x^μ on the perturbed spacetime $\mathcal{N}$. Let Q be a quantity in $\mathcal{N}$ and ${}^{(0)}Q$ be the same quantity in $\mathcal{M}$. The perturbation of Q is then the difference between these two:

$$\delta Q(y^\mu) = Q(x^\mu) - {}^{(0)}Q(y^\mu). \tag{2.8}$$

Now, let $\tilde{\mathcal{D}}$ be a different diffeomorphism which induces the coordinates $\tilde{x}^\mu$ on $\mathcal{N}$. The perturbation in this case is

$$\delta\tilde{Q}(y^\mu) = \tilde{Q}(\tilde{x^\mu}) - {}^{(0)}Q(y^\mu). \tag{2.9}$$

Hence, the two perturbations δQ and $\delta\tilde{Q}$ are related by the change in diffeomorphisms which in fact induces a coordinate transformation on $\mathcal{N}$,

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu. \tag{2.10}$$

[3]This may be considered as an active gauge transformation. For a passive approach, see [16].

In particular, we only consider infinitesimal transformations where ξ^μ is small. In this case, the coordinate transformation on $\mathcal{N}$ may be equally considered as the coordinate transformation on $\mathcal{M}$, without any reference to the perturbed spacetime, and the transformation given by (2.10) is called a *gauge transformation.* It turns out that the change of the perturbed quantity under this transformation is

$$\Delta\delta Q = \delta\tilde{Q} - \delta Q = \mathcal{L}_\xi Q, \tag{2.11}$$

where $\mathcal{L}_\xi$ denotes the Lie derivative along the vector ξ^μ. In particular, the Lie derivative of a general second rank tensor $A_{\mu\nu}$ is

$$\mathcal{L}_\xi A_{\mu\nu} = A_{\mu\nu,\lambda}\xi^\lambda + A_{\mu\lambda}\xi^\lambda{}_{,\nu} + A_{\lambda\nu}\xi^\lambda{}_{,\mu}. \tag{2.12}$$

In §2.1, we have looked at the perturbed metric tensor $\tilde{h}_{\mu\nu}$. Unfortunately, these perturbations are not uniquely determined since they will change under the gauge transformation (2.10) according to their Lie derivatives along ξ^μ. Also, as we will see in §2.3, the pure field equations are very complicated to solve explicitly. Thus we put constraints on the perturbed metric tensor by specifying ξ^μ, which reduces the number of degrees of freedom. (However, we are not really interested in the form of ξ^μ after we have made this choice, since it just relabels the coordinates and has no physical interpretations.)

As discussed by Wiltshire [17], the exact spatial homogeneity required by the FLRW model demands three restrictive conditions: (i) the existence of ideal comoving observers with synchronized clocks; (ii) the existence of constant spatial curvature hypersurfaces that are orthogonal to the geodesics of the ideal comoving observers; and (iii) that the expansion rate of such observers within such hypersurfaces is uniform. As with other perturbation methods, our approach assumes the FLRW metric as describing the *average* geometry of the universe. However, the perturbed FLRW background will not satisfy all above three conditions. Instead, we may take one condition more fundamental than the other two. For example, one of most popular choices, the synchronous gauge, best embodies the first condition. In this gauge, the constraint equations are $\tilde{h}_{00} = \tilde{h}_{0i} = 0$, which trivially reduces the degrees of freedom by two. Hence, one of the advantages of this gauge is that it makes calculations simple. Bičák, Katz and Lynden-Bell [4] consider gauge choices which best embody Mach's principle in the sense that the local inertial frames can be directly determined from the perturbed energy-momentum tensor. In fact, they consider three such conditions where, in addition, each of them imposes one of the conditions above. We will study these choices in §3.2 and §3.4.

2.3 Field equations

In general relativity, the evolution of the universe is governed by the field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}, \tag{2.13}$$

where $R_{\mu\nu}$ is the Ricci tensor which involves second derivatives of the metric. The equations (2.13) are still true in the linear perturbation framework but now the metric $g_{\mu\nu}$ is taken to be the perturbed metric ${}^{(0)}g_{\mu\nu} + h_{\mu\nu}$ as in (2.3). Hence, ignoring second order terms, (2.13) become

$$\delta R_{\mu\nu} - \frac{1}{2}{}^{(0)}g_{\mu\nu}\delta R - \frac{1}{2}h_{\mu\nu}{}^{(0)}R = 8\pi G\delta T_{\mu\nu}. \tag{2.14}$$

The explicit calculations of the perturbed field equations (2.14) are done in [4] in the coordinates (2.4) for the general case $k = \pm 1,\ 0$. We first define

$$\mathcal{K} = \frac{3}{2}\mathcal{H}\tilde{h}_{00} + \frac{1}{2}\tilde{h}'_{nn} - \nabla_i\tilde{h}_{i0}, \tag{2.15}$$

$$\mathcal{T}_j = \nabla_i\tilde{h}^T_{ij}, \tag{2.16}$$

for reasons that will be explained in the next chapter; here $\tilde{h}^T_{ij} = \tilde{h}_{ij} - \frac{1}{3}\delta_{ij}\tilde{h}_{nn}$ is the traceless part of $\tilde{h}_{ij}$, and $\mathcal{H} = \frac{a'}{a} = \dot{a}$ denotes the conformal Hubble parameter[4]. Now, separating the traceless part from the trace, we may write the perturbed field equations in terms of these variables as

$$8\pi Ga^2\delta\tilde{T}^0_0 = \frac{1}{3}\nabla^2\tilde{h}_{nn} - 2\mathcal{H}\mathcal{K} - \frac{1}{2}\nabla_n\mathcal{T}_n, \tag{2.17}$$

$$8\pi Ga^2\delta\tilde{T}^0_i = \frac{1}{2}\nabla^2\tilde{h}_{i0} + \frac{1}{6}\nabla_{ij}\tilde{h}_{j0} + \frac{2}{3}\nabla_i\mathcal{K} - \frac{1}{2}\mathcal{T}'_i, \tag{2.18}$$

$$8\pi Ga^2(\delta\tilde{T}^0_0 - \delta\tilde{T}^n_n) = \nabla^2\tilde{h}_{00} + 3a\left(\frac{\dot{a}}{a}\right)'\tilde{h}_{00} + \frac{2}{a}\left(a\mathcal{K}\right)', \tag{2.19}$$

$$\begin{aligned}8\pi Ga^2\delta\tilde{T}^T_{ij} &= -\frac{1}{2}\nabla^2\tilde{h}^T_{ij} + \frac{1}{2a^2}\left[a^2(\tilde{h}^T_{ij})'\right]' + \left(\nabla_{(i}\mathcal{T}_{j)} - \frac{1}{3}\delta_{ij}\nabla_n\mathcal{T}_n\right)\\ &-\frac{1}{a^2}\left[a^2\left(\nabla_{(i}\tilde{h}_{j)0} - \frac{1}{3}\delta_{ij}\nabla_n\tilde{h}_{n0}\right)\right]' + \frac{1}{2}\left(\nabla_{ij} - \frac{1}{3}\delta_{ij}\nabla^2\right)\left(\tilde{h}_{00} - \frac{1}{3}\tilde{h}_{nn}\right),\end{aligned} \tag{2.20}$$

where $\nabla^2 = \nabla_{nn}$ is the Laplacian. In the case of the perfect fluid, the perturbed energy-momentum tensor takes the form

$$\delta\tilde{T}^0_0 = \delta\rho, \tag{2.21}$$

$$\delta\tilde{T}^0_i = \frac{1}{4\pi Ga^2}(\mathcal{H}^2 - \mathcal{H}')(-\delta\tilde{u}_i + \tilde{h}_{i0}), \tag{2.22}$$

$$\delta\tilde{T}^0_0 - \delta\tilde{T}^n_n = \delta\rho + 3\delta p, \tag{2.23}$$

$$\delta\tilde{T}^i_{Tj} = 0. \tag{2.24}$$

As we see, the pure field equations are coupled and take a complicated form. So in general, we cannot explicitly solve for the perturbed metric tensor[5] $\tilde{h}_{\mu\nu}$. A technique which is frequently used to solve these equations is to decompose the perturbed metric into scalar, vector and tensor modes. In fact, we will use this technique to calculate density perturbations in §4.1. However, there is a certain class of gauge conditions that allows us to explicitly solve the above equations without such a decomposition. We will study such gauge conditions in the next chapter.

[4]We use the convention that an overdot $\dot{}$ denotes differentiation with respect to cosmological time t, and $'$ denotes differentiation with respect to conformal time η. For convenience, we let ∇_i to denote the spatial covariant derivative $_{;i}$.

[5]Of course, this may seem trivial as we have not yet put any constraints on the perturbed metric. However, this will still be true for an arbitrary gauge for most of the time.

Chapter 3

Codifying Mach's Principle

In this chapter we will study the recent work of Bičák, Katz and Lynden-Bell [4] on developing Machian gauges. In §3.1, we discuss the meaning of the acceleration and rotation of local inertial frames. In §3.2, 3.3, we define the Machian gauge and study the solutions of field equations. In §3.4, we discuss the possibilities of the existence of other Machian gauges.

3.1 The acceleration and rotation of local inertial frames

To consider the kinematic quantities such as the acceleration and rotation, we need to understand the difference between slicing and threading. We consider a congruence of geodesics: a set of integral curves of a vector field in $\mathcal{M}$ such that every point in $\mathcal{M}$ lies precisely on one curve. We are particularly interested in a congruence of timelike geodesics, since such integral curves may describe the worldlines of noninteracting particles. If U^μ is the vector field of the geodesic congruence, then it may define hypersurfaces everywhere orthogonal to the vector field, if the condition [18]

$$U_{[\mu}\nabla_\nu U_{\lambda]} = 0, \tag{3.1}$$

is satisfied. In this case, $\mathcal{M}$ can be foliated by hypersurfaces (or time slicings) Σ_t as before. If (3.1) is not satisfied, then hypersurfaces cannot be defined, but the geodesic congruence still threads every point in $\mathcal{M}$.

Consider a general congruence of timelike geodesics in the background FLRW spacetime manifold $\mathcal{M}$ given by

$$x^\mu = x^\mu(t, y^i), \tag{3.2}$$

where y^i is the fixed comoving coordinates and t is the parameter along the geodesic, chosen to be the cosmological time. Then the *cosmological observers* are the ones who freely travel along their worldlines described by the geodesic (3.2) for each y^i. Their normalized 4-velocity is given by

$$u^\mu = \frac{t^\mu}{(g_{\alpha\beta}t^\alpha t^\beta)^{1/2}}, \tag{3.3}$$

where $t^\mu = \partial x^\mu/\partial t|_{y_i}$. In general, u^μ does not need to satisfy (3.1), i.e., the hypersurfaces orthogonal to the vector field may not be defined. Let us take (3.3) to be the normalized timelike frame vector. To specify the frame associated with each observer, we also need to take three normalized spatial frame vectors. We need to be careful in constructing these frames since the timelike frame vector is changing with t, in such a way that the spatial frame vectors keep their orthogonality. For a given instant t, an observer sees three other nearby observers separated by[1] δy^i. Thus one may choose three linearly independent vectors that are orthogonal to u^μ:

$$\delta y^\mu_{(i)} = P^\mu{}_\nu \delta x^\nu_{(i)} = (\delta^\mu{}_\nu - u^\mu u_\nu) \left. \frac{\partial x^\nu}{\partial y^i} \right|_t \delta y^i, \tag{3.4}$$

with no summation over index i, where $P_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu$ is the projection tensor which, at each point p, projects any vector in the tangent space $T_p\mathcal{N}$ into the subspace of $T_p\mathcal{N}$ corresponding to the vectors orthonormal to u^μ. These spatial frame vectors at fixed t may be extended along the observer's worldline because they are Lie propagated along the congruence:

$$P^\mu{}_\nu \delta y^\nu_{(i);\lambda} u^\lambda = u^\mu{}_{;\nu} \delta y^\nu_{(i)}. \tag{3.5}$$

We may normalize these to give the normalized spatial frame vectors $e^\mu_{(i)}$ by

$$\delta y^\mu_{(i)} = \delta l_{(i)} e^\mu_{(i)}, \tag{3.6}$$

where $\delta l_{(i)}$ is a normalization factor. We see that $e^\mu_{(i)}$ also propagates according to (3.5). Then the *cosmological observer frame* (COF) is defined by the local frame of a cosmological observer, given by the unit orthonormal tetrad $\{u^\mu, e^\mu_{(i)}\}$. Having defined the COF, we would like to know the kinematical quantities associated with these frames. Following [18], one may decompose the derivative of the 4-velocity in the standard manner:

$$u_{\mu;\nu} = u_\nu \alpha_\mu + \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{1}{3}\theta P_{\mu\nu}, \tag{3.7}$$

where

$$\alpha_\mu = u_{\mu;\nu} u^\nu, \tag{3.8}$$
$$\omega_{\mu\nu} = \frac{1}{2} P^\rho{}_\mu P^\sigma{}_\nu (u_{\rho;\sigma} - u_{\sigma;\rho}), \tag{3.9}$$
$$\sigma_{\mu\nu} = \frac{1}{2} P^\rho{}_\mu P^\sigma{}_\nu (u_{\rho;\sigma} + u_{\sigma;\rho}) - \frac{1}{3}\theta P_{\mu\nu}, \tag{3.10}$$
$$\theta = u^\mu{}_{;\mu}, \tag{3.11}$$

represents the acceleration, vorticity, shear and the expansion rate. In the homogeneous, isotropic FLRW spacetime manifold $\mathcal{M}$, we have $u^\mu = (1, 0, 0, 0)$. In this case, we can show that all of the above quantities identically vanish except $\theta = 3H$, where $H = \frac{\dot{a}}{a}$ is the Hubble parameter. In the perturbed spacetime manifold $\mathcal{N}$ with respect to the metric (2.3), the 4-velocity $u^\mu = {}^{(0)}u^\mu + \delta u^\mu$ has the form

$$u^0 = 1 + \frac{1}{2}h_{00}, \quad u_0 = -1 + \frac{1}{2}h_{00}, \tag{3.12}$$

[1] For this example, δ denotes the virtual displacement. For the rest of the project, it will denote the first order perturbed quantities.

by the normalization condition $u^\mu u_\mu = -1$. However, there is no restriction on the spatial components u^i. In the following, we shall assume that the congruence of cosmological observers is given by $y^i =$ constant as before. In this case, we have

$$u^i = 0 \ , \quad u_i = h_{i0}. \tag{3.13}$$

In general, COFs are accelerated and rotated with respect to nonaccelerating inertial frames. (We ourselves may be considered to be cosmological observers, as we accelerate and rotate with respect to the comoving cosmic rest frame of the universe according to the peculiar motion of the earth.) Amongst those inertial frames, there is one which is momentarily at rest with respect to COF, i.e., it has the same 4-velocity at the same spacetime event. We call such a frame the *local inertial frame* (LIF). Using (3.8), we may calculate the acceleration of the global COF with respect to the LIF:

$$\alpha^0 = 0, \quad \alpha^i = -\frac{1}{2}h_{00,i} + \dot{h}_{i0}. \tag{3.14}$$

We may also calculate the vorticity and the shear, using (3.9) and (3.10),

$$\omega_{00} = \omega_{i0} = 0, \quad \omega_{ij} = \frac{1}{2}(h_{i0,j} - h_{j0,i}), \tag{3.15}$$

$$\sigma_{00} = \sigma_{i0} = 0, \quad \sigma_{ij} = \frac{1}{2}\dot{h}_{ij} - \frac{1}{6}\delta_{ij}\dot{h}_{nn} - Hh_{ij}, \tag{3.16}$$

respectively. An important remark is that none of these quantities are not exact; that is, these do not characterize the exact acceleration, vorticity and shear of the COF Also, although these expressions hold in the linear perturbation regime, they break down at some point when the higher order terms become significant. Hence, we need to be aware of the fact that our usual results in FLRW model may not hold in the linear perturbation regime. The physical quantities (3.14), (3.15), (3.16) represent the acceleration, vorticity, and shear of the COF with respect to the LIF, respectively. Hence, the acceleration, vorticity, shear of the LIF with respect to the COF are just given by the negatives of (3.14), (3.15), (3.16) respectively. In this sense, the acceleration and rotation of LIF with respect to COF are determined. To determine these quantities is to determine the perturbed metric tensor $h_{\mu\nu}$. This is in fact the whole idea behind the Machian gauge.

3.2 Machian gauge

We first discuss how Mach's principle may be incorporated in this context. Let us first recall Mach's principle: it says, "local inertial frames are determined through the distributions of energy and momentum in the Universe by some weighted averages of the apparent motions." To determine local inertial frames means to determine their acceleration and rotation. As we have seen in §3.1, we can determine these if we know the metric perturbation, $h_{\mu\nu}$.

There are two quantities that characterize the rotation: vorticity and shear. However, it turns out that we only need the vorticity $\omega_{\mu\nu}$ to completely determine the *average* rotation[2].

[2]One may argue this by considering Fermi-Walker derivatives of the unit frame vectors $e^\mu_{(i)}$. See [4] for details.

To calculate (3.14) and (3.15), we need to know the perturbed metric tensor $h_{\mu\nu}$, and in fact, only h_{00} and h_{i0}. Our goal is then to determine the perturbed metric components given the distributions of energy and momentum in the universe, represented by the perturbed energy-momentum tensor $\delta T_{\mu\nu}$. As we have seen in §2.3, the field equations are coupled, so in general it is not possible to obtain explicit solutions for $h_{\mu\nu}$. However, if we adopt some suitable conditions on the metric by working in a specific gauge, it may be possible - we will call the gauges which implement this *Machian.* These gauge conditions were first introduced by Bičák, Katz and Lynden-Bell [4], where they have considered three different constraints on spacelike hypersurfaces in addition to one constraint on the spatial coordinates. Here, we will mainly discuss one of their gauges, which may be considered as the most natural choice as it decouples the perturbed field equations the most. We discuss the other possiblities in §3.4.

We consider three conditions on the spatial metric and one condition on the hypersurface. The equation

$$\mathcal{T}_j = \nabla_i \tilde{h}^T_{ij} = 0, \tag{3.17}$$

together with the condition

$$\mathcal{K} = \frac{3}{2}\mathcal{H}\tilde{h}_{00} + \frac{1}{2}(\tilde{h}_{nn})' - \nabla_i \tilde{h}_{i0} = 0, \tag{3.18}$$

defines the *Machian gauge*[3]. The first condition is similar to the "minimal-distortion" condition studied by Smarr and York [19], in which the shear of coordinates between successive time slices is minimized. The minimal-distortion condition is equivalent to

$$D_j \dot{\tilde{\gamma}}_{ij} = 0, \tag{3.19}$$

where $\tilde{\gamma}_{ij} = (\det\gamma)^{-1/3}\gamma_{ij}$ is the conformal 3-metric induced on a given hypersurface and D_j is the covariant derivative with respect to γ_{ij}. One may show that (3.19) implies $\dot{\mathcal{T}}_i = 0$, whereas Bičák, Katz and Lynden-Bell assert the stronger condition (3.17). This gauge condition has not been studied in detail before, and they explain it as: *the spatial coordinates are restricted on an initial slice and this restriction is then maintained by the original condition.* However, (3.17) certainly does not imply (3.19) and may not even have any relation with the shear. Thus, the physical validity of (3.17) still need to be examined to answer the question of whether the Machian gauge gives physically valid foliations of spacetime.

The condition (3.18) characterizes the "uniform-Hubble-constant hypersurfaces" studied in the classic work of Bardeen [2]. To understand its meaning, let us consider the unit timelike vector field $\tilde{n}^\mu$, orthogonal to the hypersurfaces foliating the perturbed spacetime manifold $\mathcal{N}$. We have

$$\tilde{n}^0 = \frac{1}{2a}(2 + \tilde{h}_{00}), \quad \tilde{n}^i = \frac{1}{a}\tilde{h}_{i0}, \tag{3.20}$$

and the contravariant components are given by

$$\tilde{n}_0 = \frac{a}{2}(-2 + \tilde{h}_{00}), \quad \tilde{n}_i = 2a\tilde{h}_{i0}. \tag{3.21}$$

[3]They refer to this choice as Mach 1 gauge as they study a number of Machian gauges. This is the only gauge we review and so we will simply refer to it as *the* Machian gauge.

Although $\tilde{n}^\mu$ is a first-order quantity, it is exact, i.e., it is really orthogonal to the hypersurfaces. Hence, the expressions (3.20) and (3.21) satisfy (3.1), the condition needed for $\tilde{n}^\mu$ to be a vector field orthogonal to the hypersurfaces. Now the expansion rate $\tilde{\theta} = \tilde{n}^\mu{}_{;\mu}$ of the congruence of timelike curves with their normal vector field $\tilde{n}^\mu$, is given by

$$\tilde{\theta} = 3H - \frac{1}{a}\left(\frac{3}{2}\mathcal{H}\tilde{h}_{00} + \frac{1}{2}\tilde{h}'_{nn} - \nabla_i \tilde{h}_{i0}\right). \tag{3.22}$$

Comparing with $\tilde{\theta} = {}^{(0)}\tilde{\theta} + \delta\tilde{\theta}$, we notice our gauge condition (3.18) is the same as $\delta\tilde{\theta} = 0$. Hence, (3.18) just means that the expansion rate of the perturbed spacetime is the same as that of the background; hence the name uniform-Hubble-expansion. Geometrically, this says that the extrinsic curvature of each hypersurface foliating the perturbed spacetime is the same as that of the hypersurface foliating the background spacetime. The global behaviour of the solutions of the field equations in this gauge is discussed in detail in [20].

Using (3.17) and (3.18), we can re-express the perturbed field equations (2.21)-(2.24) in the Machian gauge as

$$\nabla^2 \tilde{h}_{nn} = 24\pi G a^2 \delta\tilde{T}_{00}, \tag{3.23}$$

$$\nabla^2 \tilde{h}_{i0} + \frac{1}{3}\nabla_{ij}\tilde{h}_{j0} = 16\pi G a^2 \delta\tilde{T}_{0i}, \tag{3.24}$$

$$\nabla^2 \tilde{h}_{00} + 3a^2\left(\frac{\dot{a}}{a}\right)^{\cdot}\tilde{h}_{00} = 8\pi G a^2(\delta\tilde{T}_{00} - \delta\tilde{T}_{nn}), \tag{3.25}$$

$$\begin{aligned} \nabla^2 \tilde{h}^T_{ij} - \frac{1}{a}\left(a^3\dot{\tilde{h}}^T_{ij}\right)^{\cdot} - \frac{1}{a^2}\left[a^2\left(\nabla_{(i}\tilde{h}_{j)0} - \frac{1}{3}\delta_{ij}\nabla_i\tilde{h}_{i0}\right)\right]' \\ + \frac{1}{2}\left(\nabla_{ij} - \frac{1}{3}\delta_{ij}\nabla^2\right)\left(\tilde{h}_{00} - \frac{1}{3}\tilde{h}_{nn}\right) = -16\pi G a^2 \delta\tilde{T}^T_{ij}, \end{aligned} \tag{3.26}$$

The above equations need to be carefully considered. Suppose we are given the form of the perturbed energy-momentum tensor $\delta\tilde{T}_{\mu\nu}$. Then we immediately know the right hand side of the equations (3.23)-(3.26). Equations (3.23) and (3.25) are decoupled equations for $\tilde{h}_{nn}$ and $\tilde{h}_{00}$ respectively, so we can solve them. By knowing these two, we can subsequently solve (3.24) for $\tilde{h}_{i0}$. Lastly, we can solve (3.26) for $\tilde{h}^T_{ij}$, and we may combine this with $\tilde{h}_{nn}$ to recover $\tilde{h}_{ij}$. Thus, we can solve for the full perturbed metric $\tilde{h}_{\mu\nu}$. Consequently, we can calculate the α_μ and $\omega_{\mu\nu}$ from (3.14) and (3.15). Hence, the Machian gauge has a unique property that we can determine the acceleration and rotation of the LIF with respect to COF directly from the perturbed energy-momentum tensor $\delta\tilde{T}_{\mu\nu}$.

3.3 Gauge fixing and solving the field equations

In §3.2 the Machian gauge is defined by (3.17) and (3.18). However, we would like $\mathcal{T}_i$ and $\mathcal{K}$ to transform under the gauge transformation (2.10) in such a way that they remain zero in the new coordinate system; that is, we require $\Delta\mathcal{T}_i = \mathcal{L}_\xi\mathcal{T}_i = 0$ and $\Delta\mathcal{K} = \mathcal{L}_\xi\mathcal{K} = 0$. This is possible because we have freedom in specifying the vector ξ^μ, which is called *gauge fixing*. The vector ξ^μ can be determined by solving the equations $\Delta\mathcal{T}_i = 0$ and $\Delta\mathcal{K} = 0$. We have

$$\Delta\mathcal{T}_i = -\nabla^2\xi^i - \frac{1}{3}\nabla_{ij}\xi^j = 0, \tag{3.27}$$

$$\Delta\mathcal{K} = -\frac{1}{a}\nabla^2\xi^0 - 3a\left(\frac{\dot{a}}{a}\right)^{\cdot}\xi^0 = 0, \tag{3.28}$$

where $\nabla_{ij} \equiv \nabla_i \nabla_j$. Note that (3.27) and (3.28) is the homogeneous equation for ξ^i and ξ^0, respectively. To solve (3.27), consider the conformal Killing equation in 3-dimensional Euclidean space $\mathbb{R}^3$ [21]:

$$\nabla_i \psi_j + \nabla_j \psi_i = \frac{2}{3} \delta_{ij} \nabla_n \psi^n. \tag{3.29}$$

There are 10 independent solutions of (3.29): 6 pure Killing vectors (satisfying the pure Killing equation $\nabla_i \psi_j + \nabla_j \psi_i = 0$), and 4 conformal Killing vectors. The remaining 6 Killing vectors consists of 3 translational and 3 rotational Killing vectors. In the coordinates $x^\mu = (t, x, y, z)$ as in (2.3), the translational Killing vectors are

$$\zeta^i_{(1)} = (x, 0, 0),\ \zeta^i_{(2)} = (0, y, 0),\ \zeta^i_{(3)} = (0, 0, z), \tag{3.30}$$

and the rotational Killing vectors are

$$\zeta^i_{(4)} = (0, -z, y),\ \zeta^i_{(5)} = (z, 0, -x),\ \zeta^i_{(6)} = (-y, x, 0). \tag{3.31}$$

Finally, the conformal Killing vectors are

$$\begin{aligned} \chi^i_{(1)} = (x, y, z),\ \chi^i_{(2)} = (\frac{1}{2} r^2 - x^2, -xy, -xz), \\ \chi^i_{(3)} = (-xy, \frac{1}{2} r^2 - y^2, -yz),\ \chi^i_{(4)} = (-xz, -yz, \frac{1}{2} r^2 - z^2), \end{aligned} \tag{3.32}$$

where $r^2 = x^2 + y^2 + z^2$. Now, taking the divergence ∇_j of (3.29), we obtain (3.27). Hence, the general solution of (3.27) is given by a linear combination of conformal Killing vectors (3.30)-(3.32). However, Bičák, Katz and Lynden-Bell [4] proved that there exists no bounded solution of (3.27) other than the translational Killing vectors[4]. Hence, the general bounded solution of (3.27) is

$$\xi^i = \Sigma^3_{I=1} \alpha_{(I)} \zeta^i_{(I)}, \tag{3.33}$$

where $\alpha_{(I)},\ I = 1, 2, 3$, are arbitrary functions of time.

Now we shall consider the residual gauge freedom of $\mathcal{K}$, which is given by the solutions of

$$\nabla^2 \xi^0 + 3a^2 \left(\frac{\dot{a}}{a}\right)^{\cdot} \xi^0 = 0. \tag{3.34}$$

Rearranging and multiplying (3.34) by ξ^0 and integrating over a spatial domain $\mathcal{D}$ on the hypersurface Σ_t of constant t gives

$$\begin{aligned} \int_{\mathcal{D}} \xi^0 \nabla^2 \xi^0 dV &= \int_{\partial \mathcal{D}} \xi^0 \nabla_i \xi^0 dS^i - \int_{\mathcal{D}} (\nabla_i \xi^0)^2 dV \\ &= - \int_{\mathcal{D}} 3a^2 \left(\frac{\dot{a}}{a}\right)^{\cdot} (\xi^0)^2 dV, \end{aligned} \tag{3.35}$$

where $dV = dx\, dy\, dz$ is a comoving volume element, and in the second step we have used integration by parts. If we take $\mathcal{D}$ to be all space, the integral over the boundary vanishes because of the boundary conditions on ξ^0 (since, as with ξ^i, we only want bounded solutions). Hence, we have

$$\int_{\mathcal{D}} (\nabla_i \xi^0)^2 dV = 3a^2 \left(\frac{\dot{a}}{a}\right)^{\cdot} \int_{\mathcal{D}} (\xi^0)^2 dV. \tag{3.36}$$

[4]This is true for $\mathbb{R}^3$. For the hyperbolic $\mathbb{H}^3$, there are no bounded solutions at all. For the spherical $\mathbb{S}^3$, all of the conformal Killing vectors are bounded.

However,

$$3a^2\left(\frac{\dot{a}}{a}\right)^{\cdot} = 3a^2\dot{H} = -12\pi Ga^2(\rho+p) = -12\pi Ga^2\rho(1+w), \tag{3.37}$$

where $p = w\rho$, $-1 \leq w \leq 1$ to satisfy the dominant energy condition. Hence, the right hand side of (3.36) is nonpositive (and 0 if $w = -1$) whereas the left hand side is nonnegative. Thus, the only possible solution of (3.28) is $\xi^0 = 0$ when $w \neq -1$ and $\xi^0 = \xi^0(t)$ when $w = -1$. In summary, (in the model where $w \neq -1$), the gauge transformation is given by

$$\xi^\mu = \left(0, \Sigma_{I=1}^3 \alpha_{(I)}\zeta^i_{(I)}\right), \tag{3.38}$$

and fixes the Machian gauge.

Having gone through its gauge fixing procedure, we shall now attempt to solve the field equations (3.23)-(3.26) for $\tilde{h}_{\mu\nu}$ assuming the source $\delta\tilde{T}_{\mu\nu}$ is given. The solution of (3.23) may be obtained by the use of Green's function (see Appendix B for details):

$$\tilde{h}_{nn} = -6Ga^2 \int \frac{\delta\tilde{T}_{00}}{|x^i - x'^i|} dV. \tag{3.39}$$

Note that applying ∇_i on (3.24) and defining $\mathcal{P} = \nabla_i\tilde{h}_{i0}$, we have

$$\nabla^2\mathcal{P} = 12\pi Ga^2\nabla_i\delta\tilde{T}_{0i}, \tag{3.40}$$

where we have substituted $\mathcal{P} = \frac{3}{2}\mathcal{H}\tilde{h}_{00} + \frac{1}{2}(\tilde{h}_{nn})'$ from the gauge condition (3.28). Now, (3.40) takes the same form as (3.23), so the solution can be correspondingly obtained:

$$\mathcal{P} = -3Ga^2 \int \frac{\nabla_i\delta\tilde{T}_{0i}}{|x^i - x'^i|} dV. \tag{3.41}$$

To determine $\tilde{h}_{i0}$, rearrange (3.24) so that we have

$$\nabla^2\tilde{h}_{i0} = 16\pi Ga^2\delta\tilde{T}_{0i} - \frac{1}{3}\nabla_i\mathcal{P}. \tag{3.42}$$

With $\mathcal{P}$ known, this is again just the Laplacian of $\tilde{h}_{i0}$ with the source on the right hand side given, so we have

$$\tilde{h}_{i0} = \frac{Ga^2}{3} \int \frac{\nabla_i\mathcal{P} - 12\delta\tilde{T}_{0i}}{|x^i - x'^i|} dV. \tag{3.43}$$

Similarly, $\tilde{h}_{00}$ can be determined by directly solving (3.25), or by solving (3.28) with $\tilde{h}_{nn}$ and $\tilde{h}_{i0}$ known. Finally, $\tilde{h}^T_{ij}$ can be determined by (3.26) and combined with $\tilde{h}_{nn}$ to recover $\tilde{h}_{ij}$. Hence, we have determined the full perturbed metric tensor $\tilde{h}_{\mu\nu}$. In fact, to calculate the acceleration (3.14) and vorticity (3.15) of local inertial frames, we only need to know $\tilde{h}_{00}$ and $\tilde{h}_{i0}$. In summary, given $\delta\tilde{T}_{\mu\nu}$, we may obtain explicit solutions for $\tilde{h}_{\mu\nu}$ and consequently determine α_μ and $\omega_{\mu\nu}$.

3.4 Potential Machian gauges

The Machian gauge we have discussed so far involved three spatial gauge conditions (3.27) and one condition on the hypersurfaces (3.28). In this gauge, we were able to explitly solve the perturbed field equations for the perturbed metric components. In their paper, Bičák, Katz and Lynden-Bell in fact discuss two more gauges which also implement this. With the spatial gauge conditions unchanged, the other conditions on the hypersurfaces they consider are: "uniform-intrinsic-curvature" and "minimal-shear" conditions. In general relativity we have two notions of curvature: extrinsic and intrinsic. The first is associated with the geometry of the way the hypersurfaces are foliated through the spacetime whereas the latter is an intrinsic property of the hypersurfaces. For the Machian gauge, the uniform-Hubble-expansion condition in fact is the same as $\delta K = 0$ where $K = K^{\mu}{}_{\mu}$ is the trace of the extrinsic curvature tensor $K_{\mu\nu} = \frac{1}{2}\mathcal{L}_{\tilde{n}} P_{\mu\nu}$, where now $P_{\mu\nu}$ is the first fundamental form which projects vectors in $T_p\mathcal{N}$ to the hypersurface, orthogonal to $\tilde{n}^{\mu}$. Hence, (3.28) often is refered to as the uniform-extrinsic-curvature or constant mean curvature (CMC) condition. In contrast, a uniform-intrinsic-curvature condition is given by

$$\delta\mathcal{R} = -\frac{2}{3a}\nabla^2\tilde{h}_{nn} + \frac{1}{a^2}\nabla_n\mathcal{T}_n = 0, \tag{3.44}$$

where $\mathcal{R} = P^{\mu\nu}\mathcal{R}^{\lambda}{}_{\mu\lambda\nu}$ is the intrinsic 3-scalar curvature of the hypersurface, and $\mathcal{R}^{\rho}{}_{\sigma\mu\nu}$ satisfies Gauss's equation:

$$\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = P^{\rho}{}_{\alpha}P^{\beta}{}_{\sigma}P^{\gamma}{}_{\mu}P^{\delta}{}_{\nu}R^{\alpha}{}_{\beta\gamma\delta} + K^{\rho}{}_{\mu}K_{\sigma\nu} - K^{\rho}{}_{\nu}K_{\sigma\mu}. \tag{3.45}$$

Note that (3.44) combined with (3.17) just requires $\nabla^2\tilde{h}_{nn} = 0$. Geometrically, this says the the intrinsic curvature of each hypersurface foliating the perturbed spacetime is the same as that of the hypersurface foliating the background spacetime. The minimal-shear hypersurface condition is given by $\nabla_{ij}K_{ij} = 0$, which is equivalent to

$$\nabla_{ij}\delta\tilde{\sigma}_{ij} = -\frac{2}{3}a\nabla^2\nabla_i\tilde{h}_{i0} + \frac{1}{2}a\nabla_n\mathcal{T}'_n = 0. \tag{3.46}$$

With the definition $\mathcal{P}$ we had before, (3.46) combined with (3.17) just requires $\nabla^2\mathcal{P} = 0$. We may apply similar techniques as in §3.3 to explicitly obtain solutions of the field equations for both gauge conditions. The details are outlined in [4]. Hence, we may correspondingly determine the acceleration (3.14) and (3.15), implementing the Machian nature. Also, both gauge conditions have a natural geometrical meaning defined on the hypersurfaces.

Many of the gauge conditions we have discussed put constraints on the kinematic quantities associated with ideal observers. In particular, the extrinsic curvature $K_{\mu\nu}$ can be decomposed into acceleration, expansion and shear similarly to the general velocity gradient of a geodesic congruence in (3.7). We might therefore ask whether there are any natural gauge conditions directly associated with the kinematic invariants of geodesic congruences. (Those associated with expansion and shear have already been considered in [4].) An attempt was made to find a new gauge for the case of vorticity, as presented in Appendix A. However, it was then recalled that although vorticity in non-zero in general congruences, it must necessarily vanish for congruences orthogonal hypersurfaces.

Efforts to consider vorticity, as in Appendix A, may not be entirely fruitless, however, if we consider the more general averaging problem. The real universe is not globally hyperbolic, and actual galaxies in the universe do possess vorticity. Thus the mathematical limitations imposed by choosing a globally hyperbolic slicing of spacetime geometry may be too restrictive for actual physics once we go beyond the perturbative FLRW regime.

There are questions to be answered when defining a new gauge condition: Can we associate this condition with an actual geometrical entity describing the geometry of foliations of hypersurfaces, which is the extrinsic curvature tensor $K_{\mu\nu}$? Then what are its physical interpretations? We need to examine these questions before we can convince ourselves that it is indeed a useful gauge to consider. One should be always open for the possibility of new gauges; however, we must understand their physical interpretations and under what assumption they are valid before we progress. For this reason, defining a useful gauge is a difficult problem.

3.5 Discussion

One question needs to be answered before we progress: What does the gauge condition (3.17) physically imply? In fact, all of the Machian gauges considered by Bičák, Katz and Lynden-Bell imposed it as the condition on the spatial metric. However, we must understand its physical and geometrical meaning and make sure it is valid in order to consider the hypersurfaces defined by such a condition as physically valid foliations of spacetime. For this reason, one may wish to search for other Machian gauges without involving (3.17). However, as mentioned in the last section, this is a difficult problem, so it is best to study this gauge and test it in various ways before attempt to formulate a new Machian gauge.

There is an important problem in general relativity: that there is no preferred foliation of spacetime by hypersurfaces, or the "fitting problem" for cosmology [11]. Physics does not depend on the choice of the coordinates but one must always work in specific coordinates in order to do physics. Having defined the coordinates on the background spacetime $\mathcal{M}$, we can define the corresponding coordinates on the perturbed spacetime $\mathcal{N}$, connected by a map, or a diffeomorphism. As seen in §2.2, this leaves a gauge freedom in choosing the coordinates on $\mathcal{N}$, or choosing the vector ξ^{μ}. Hence, there are various ways to define a mapping between the perturbed spacetime manifold and the background spacetime manifold. However, each gauge puts constraints on the foliations of the hypersurfaces with its own geometrical meaning, so there is no unique and best way to do this. However, there might be a preferred choice for describing the *average* dynamics, since there appears to be an average notion of homogeneity, with a preferred comoving cosmic rest frame.

Can this idea be applied to inhomogeneous cosmology? The notion of averaging is important in the study of inhomogeneous cosmology, and one simple way of averaging is proposed by Buchert in [13]. This Buchert average has been used in various papers, including [27], [22]. A recent paper by Morita *et al.* [23] considers the relative information entropy of an inhomogeneous universe in the synchronous gauge. So one may repeat this with the Machian gauge to see its effect. However, the problem is that the Buchert average is only well-defined for the synchronous gauge. Hence, one needs to define a new averaging formalism before we can apply the result to inhomogeneous cosmology. This is indeed a difficult problem and is left to future work.

Chapter 4

Applications

In the previous two chapters, we have dealt with the full metric perturbations $\tilde{h}_{\mu\nu}$, and examined techniques to solve for them. The motivation was to emphasize the fact that we can determine the acceleration (3.14) and rotation (3.15) of the local inertial frames. However, in practice, we do not want to solve for the full metric perturbation for the two reasons: (i) it is difficult to solve, and (ii) only certain part of the perturbation metric leads to growth of inhomogeneities. Instead, as first considered by Lifshitz [1], we decompose the metric perturbation into scalar, vector and tensor perturbations, and it turns out that the vector modes decay while the tensor modes do not couple to density and pressure inhomogeneities. Hence, it is the scalar perturbations that lead to the growth of density perturbations and the structure formation that cosmologists are interested in. For this reason, we will study the scalar perturbations in the Machian gauge. In §4.1, we briefly review the general procedure of scalar, vector and tensor decomposition of the metric following Bertschinger [24], [25]. In §4, we calculate the density perturbations by solving the field equations in the matter- and radiation-dominated universes.

4.1 Scalar, vector and tensor decomposition

We may write the line element (2.4) as follows:

$$ds^2 = a^2 \left[-(1+2\psi)d\eta^2 + 2w_i d\eta dx^i + [(1-2\phi)\delta_{ij} + S_{ij}]dx^i dx^j \right], \tag{4.1}$$

where $\tilde{h}_{00} = -2\psi$, $\tilde{h}_{i0} = w_i$, and $\tilde{h}_{ij} = 2\delta_{ij}\phi + S_{ij}$. Without loss of generality, we may impose $S_{ii} = 0$, as the trace can be absorbed into ϕ. Here, all of ψ, ϕ, w_i, S_{ij} are 3-tensors (of rank 0, 0, 1, 2 respectively). Now, according to the Helmholtz theorem in ordinary vector calculus [26], for any 3-vector field w_i, there exists a scalar w and a solenoidal vector field $w_i^\perp$ such that

$$w_i = \nabla_i w + w_i^\perp. \tag{4.2}$$

Hence, a part of w_i can be obtained from a scalar, and only $w_i^\perp$ corresponds to pure vector perturbation. Similarly, a 3-tensor S_{ij} can be decomposed as

$$S_{ij} = S_{ij}^\parallel + S_{ij}^\perp + S_{ij}^{\mathrm{T}}, \tag{4.3}$$

satisfying

$$S_{ij}^{\parallel} = (\nabla_{ij} - \frac{1}{3}\delta_{ij}\nabla^2)S, \tag{4.4}$$

$$S_{ij}^{\perp} = \frac{1}{2}(\nabla_i S_j^{\perp} + \nabla_j S_i^{\perp}), \tag{4.5}$$

$$\nabla_i S_{ij}^{\mathrm{T}} = 0 \qquad S_{ii}^{\mathrm{T}} = 0. \tag{4.6}$$

Hence, $S_{ij}^{\parallel}$ and $S_{ij}^{\perp}$ corresponds to pure scalar and vector perturbations respectively, and S_{ij}^{T} corresponds to pure tensor perturbations. Consequently, a pure scalar metric perturbation can be written as

$$\tilde{h}_{00} = -2\psi, \quad \tilde{h}_{i0} = -\nabla_i B, \quad \tilde{h}_{ij} = -2\delta_{ij}\phi + \nabla_{ij}E, \tag{4.7}$$

where we have defined $B = w$ and $\nabla_{ij}E = S_{ij}^{\parallel}$ with $\nabla^2 E = 0$.

4.2 Examples

For the scalar mode, the gauge condition (3.17) is naturally incorporated since[1] $\mathcal{T}_j = \nabla_i(\nabla_{ij}E) = \nabla_j(\nabla^2 E) = 0$. Using the perturbed metric (4.7), the field equations (3.23)-(3.26) for the perfect fluid become

$$\nabla^2\psi = 4\pi G a^2 \delta\rho, \tag{4.8}$$

$$\left(\frac{1}{3}\nabla^2 + a^2\kappa(a\ddot{a} - \dot{a}^2)\right)\nabla_i B = 8\pi G a(a\ddot{a} - \dot{a}^2)\delta u_i, \tag{4.9}$$

$$\nabla^2\phi + 3a^2\left(\frac{\dot{a}}{a}\right)^{\cdot}\phi = 4\pi G a^2(\delta\rho + 3\delta p), \tag{4.10}$$

$$\nabla^2\phi - \nabla^2\psi + 2\dot{a}\nabla^2 B + a\nabla^2\dot{B} = 0, \tag{4.11}$$

and the gauge condition $\mathcal{K} = 0$ gives

$$\dot{a}\phi + a\dot{\psi} + \frac{1}{3}\nabla^2 B = 0. \tag{4.12}$$

The background is described by the Friedmann equation

$$H^2 = \left(\frac{\mathcal{H}}{a}\right)^2 = \frac{8\pi G}{3}{}^{(0)}\rho. \tag{4.13}$$

Using the background equation (4.13) and defining density contrast $\delta = \frac{\delta\rho}{{}^{(0)}\rho}$ we get

$$\nabla^2\psi = \frac{3}{2}\mathcal{H}^2\delta, \tag{4.14}$$

$$\left(\frac{1}{3}\nabla^2 + a^2\kappa(a\ddot{a} - \dot{a}^2)\right)\nabla_i B = 8\pi G a(a\ddot{a} - \dot{a}^2)\delta u_i, \tag{4.15}$$

$$\nabla^2\phi + 3a^2\left(\frac{\dot{a}}{a}\right)^{\cdot}\phi = \frac{3}{2}\mathcal{H}^2(1 + 3w)\delta, \tag{4.16}$$

$$\nabla^2\phi - \nabla^2\psi + 2\dot{a}\nabla^2 B + a\nabla^2\dot{B} = 0. \tag{4.17}$$

[1]However, this is only true in flat geometry, since covariant derivatives do not commute in non-flat geometry.

We have in total five equations (4.12), (4.14)-(4.17) and five variables $(\delta, \psi, \phi, B, \delta u^i)$. However, the density constrast δ is the quantity we would like to investigate to study structure formation. Hence, to obtain the solution for δ we will only need four coupled equations (4.12), (4.14), (4.16), (4.17).

In the following, we will assume a one-component fluid to simplify the calculations, (i.e., the energy density ρ equals one of ρ_m, ρ_r and ρ_Λ for matter, radiation and dark energy, respectively). This is a good approximation for certain epochs when one form of energy is dominant. The dark energy-dominated era does not come about until late epochs when the universe was about 5 billion years old, so in particular, we will consider matter- and radiation-dominated universes.

4.2.1 Matter-dominated universe

In the matter-dominated universe[2], we may express the scale factor a as

$$a = \left(\frac{\eta}{\eta_0}\right)^2. \tag{4.18}$$

In general, even though we are only considering scalar perturbations for simplicity, it is still difficult to explicitly solve the given field equations. However, there is another symmetry we can exploit: that the equations are symmetric under spatial translations. In a spatially flat geometry, the the general solution can be written as the superposition of plane waves $e^{-i\mathbf{k}\cdot\mathbf{x}}$ which are eigenfunctions of the Laplacian ∇^2. Thus, it is natural to work with the Fourier modes defined by

$$\phi(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \phi_{\mathbf{k}}(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}}, \tag{4.19}$$

and so on for other variables where $\mathbf{k}$ is the wavevector of wavelength $2\pi a/k$. In this case, we may obtain differential equations for the Fourier amplitudes where we replace the spatial derivatives ∇_i with ik. The new differential equations then only depend on the magnitude k not the direction of $\mathbf{k}$. The equations (4.12), (4.14), (4.16), (4.17) in terms of the Fourier amplitudes are

$$\frac{k^2\eta^2}{6}\psi_{\mathbf{k}} = -\delta_{\mathbf{k}}, \tag{4.20}$$

$$\frac{k^2\eta^2}{6}\phi_{\mathbf{k}} + 3\phi_{\mathbf{k}} = -\delta_{\mathbf{k}}, \tag{4.21}$$

$$\frac{2}{\eta}\phi_{\mathbf{k}} + \psi'_{\mathbf{k}} - \frac{k^2}{3}B_{\mathbf{k}} = 0, \tag{4.22}$$

$$\phi_{\mathbf{k}} - \psi_{\mathbf{k}} + \frac{4}{\eta}B_{\mathbf{k}} + B'_{\mathbf{k}} = 0. \tag{4.23}$$

We have four coupled equations for $(\delta_{\mathbf{k}}, \psi_{\mathbf{k}}, \phi_{\mathbf{k}}, B_{\mathbf{k}})$. We may combine (4.20) with (4.21) and combine (4.22) with (4.23) to eliminate $\delta_{\mathbf{k}}$ and $B_{\mathbf{k}}$, respectively. Subsequently, these two equations can be combined to give an equation for $\phi_{\mathbf{k}}$

$$\left(\frac{1}{6} + \frac{3}{k^2\eta^2}\right)\phi''_{\mathbf{k}} + \frac{1}{\eta}\phi'_{\mathbf{k}} - \frac{6}{k^2\eta^4}\phi_{\mathbf{k}} = 0, \tag{4.24}$$

[2]This is equivalent to the Einstein-de Sitter model, i.e., $a \propto t^{2/3} \propto \eta^2$.

which is a second-order linear differential equation we can solve. We may solve the above using Maple to obtain

$$\phi_{\mathbf{k}}(\eta) = \frac{1}{(k^2\eta^2+18)^2}\left[\frac{C_1}{\eta} + C_2(30\eta^2 + k^2\eta^4)\right], \tag{4.25}$$

where C_1 and C_2 are integration constants depending on $\mathbf{x}$. We now insert the above to (4.21) and obtain the following expression for the density contrast

$$\delta_{\mathbf{k}}(\eta) = \frac{1}{(k^2\eta^2+18)^2}\left[C_1\left(\frac{3}{\eta} + \frac{k^2\eta}{6}\right) + C_2\left(\frac{k^4\eta^6}{180} + \frac{k^2\eta^4}{15} + 3\eta^2\right)\right]. \tag{4.26}$$

We have two independent solutions: one with $C_1 = 0$ and one with $C_2 = 0$. Consider the case where $C_1 = 0$ and $C_2 \neq 0$. This corresponds to a growing mode solution since we have η^2 behaviour in the long term, whereas the case with $C_1 \neq 0$ and $C_2 = 0$ corresponds to a decaying mode because of the η^{-3} long-term behaviour. Consider the growing mode solution

$$\delta^+_{\mathbf{k}}(\eta) = \frac{\eta^2}{(k^2\eta^2+18)^2}\left[\frac{k^4\eta^4}{180} + \frac{k^2\eta^2}{15} + 3\right]. \tag{4.27}$$

The Hubble length $\mathcal{H}^{-1}$ defines a cosmological particle horizon which corresponds to the maximum size of the universe we can observe. However, for an accelerating universe, the Hubble length does not necessarily define a cosmological event horizon, and there may be particles beyond the particle horizon - the superhorizon. Hence, we examine the behaviour of the density contrast for short-wavelength limit $k\eta \gg 1$ and long-wavelength limit $k\eta \ll 1$, i.e., we are well-inside and beyond the particle horizon respectively. We have

$$\delta^+_{\mathbf{k}}(\eta) = \begin{cases} \frac{\eta^2}{180}, & k\eta \gg 1 \\ \frac{\eta^2}{108}, & k\eta \ll 1 \end{cases} \tag{4.28}$$

So the growing mode of the density contrast grows with $\eta^2 \sim t^{2/3}$ for both short- and long-wavelength limit, i.e., $\delta^+_{\mathbf{k}}(t) \propto a$. Now consider the decaying mode solution

$$\delta^-_{\mathbf{k}}(\eta) \equiv \frac{1}{(k^2\eta^2+18)^2}\left[\frac{3}{\eta} + \frac{k^2\eta}{6}\right]. \tag{4.29}$$

For the two limits, we have

$$\delta^-_{\mathbf{k}}(\eta) = \begin{cases} 0, & k\eta \gg 1 \\ \frac{1}{108\eta}, & k\eta \ll 1 \end{cases} \tag{4.30}$$

So the decaying mode solution is zero for the long-wavelength limit and decays with $\eta^{-1} \sim t^{-1/3}$, i.e., $\delta^-_{\mathbf{k}} \propto a^{-1}$. The plots of (4.27) and (4.29) are given in Figure 4.1. As we expect, the density constrast grows in the growing mode and decays rapidly in the decaying mode. We need to correct the overall normalization factor in order to compare with other models but the purpose of these plots is to illustrate the general behaviour of density fluctuations in the Machian gauge.

To summarize, the density perturbations in the Machian gauge have the two usual solutions: growing and decaying modes. The density perturbation in the growing mode grows proportional to a in both short- and long-wavelength limits. Physically, the growing mode

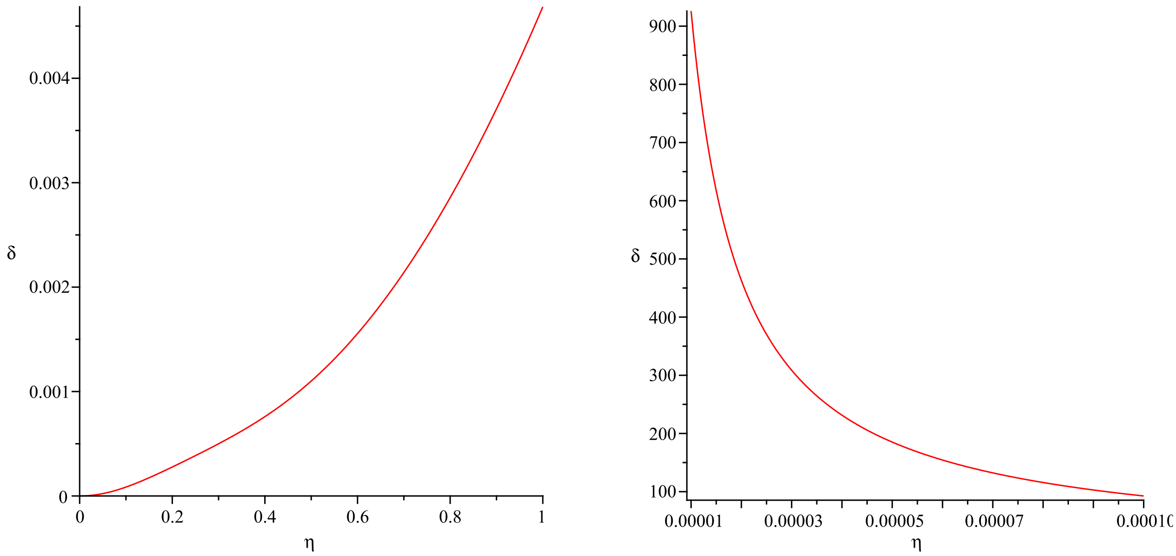

Figure 4.1: The density contrast, $\delta_{\mathbf{k}}$, for the matter dominated universe, is plotted against conformal time for the slice $k = 1$ and the initial condition $C_1 = C_2 = 1$. The left panel shows the growing mode $\delta_{\mathbf{k}}^+$ and the right panel shows the decaying mode $\delta_{\mathbf{k}}^-$.

represents the growth of (matter) inhomogeneities. When these inhomogeneities become large enough, i.e., when over- and underdense regions start to form, gravitational instability occurs, which allows structures to form. For the decaying mode, the density perturbation is either zero or proportional to a^{-1} in the long- and short-wavelength limit, respectively. Physically, these solutions represent perturbations with initial radial peculiar velocities such that they 'undo' the self-gravitational attraction of the overdense regions.

4.2.2 Radiation dominated universe

In the radiation-dominated universe, we may express the scale factor a as

$$a = \frac{\eta}{\eta_0}. \tag{4.31}$$

Again, we may work in Fourier space with Fourier amplitudes defined by (4.19). Then with the scale factor (4.31), the equations (4.12), (4.14), (4.16), (4.17) for Fourier modes take the form

$$\frac{k^2\eta^2}{3}\psi_{\mathbf{k}} = -\frac{1}{2}\delta_{\mathbf{k}}, \tag{4.32}$$

$$\frac{k^2\eta^2}{3}\phi_{\mathbf{k}} + 2\phi_{\mathbf{k}} = -\delta_{\mathbf{k}}, \tag{4.33}$$

$$\frac{1}{\eta}\phi_{\mathbf{k}} + \psi_{\mathbf{k}}' - \frac{k^2}{3}B_{\mathbf{k}} = 0, \tag{4.34}$$

$$\phi_{\mathbf{k}} - \psi_{\mathbf{k}} + \frac{2}{\eta}B_{\mathbf{k}} + B_{\mathbf{k}}' = 0. \tag{4.35}$$

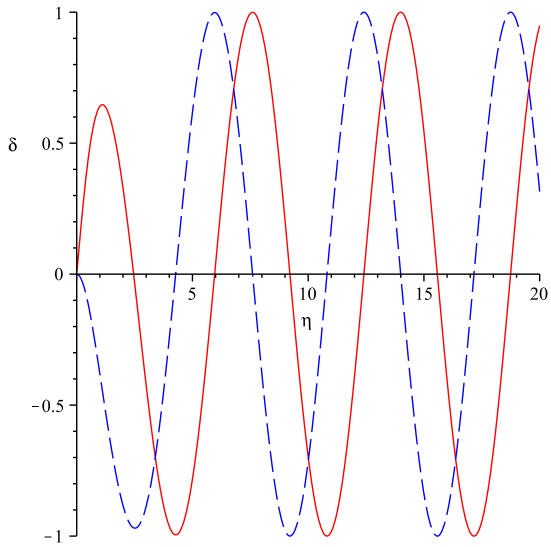

Figure 4.2: The density contrast, $\delta_{\mathbf{k}}$, for the radiation dominated universe, is plotted against conformal time for the slice $k = 1$. The solid line represents the solution with $C_1 = 1$, $C_2 = 0$ and the dashed line represents represents the solution with $C_1 = 0$, $C_2 = 1$.

Using the same techniques as in §4.2.1, the above four equations can be combined to give an equation for $\phi_{\mathbf{k}}$

$$\left(\frac{1}{2} + \frac{3}{k^2\eta^2}\right)\phi''_{\mathbf{k}} + \left(2 - \frac{6}{k^2\eta^2}\right)\frac{\phi'_{\mathbf{k}}}{\eta} + \left(\frac{k^2}{6} + \frac{6}{k^2\eta^4}\right)\phi_{\mathbf{k}} = 0 \ . \tag{4.36}$$

Solving the above using Maple gives

$$\phi_{\mathbf{k}}(\eta) = \frac{\eta}{(\omega^2\eta^2 + 2)^2}\left[C_1(\omega\eta\cos\omega\eta - 2\sin\omega\eta) + C_2(\omega\eta\sin\omega\eta + 2\cos\omega\eta)\right], \tag{4.37}$$

where we have defined[3] $\omega = k/\sqrt{3}$. Now we may substitute (4.37) into (4.33) to obtain the expression for the density contrast

$$\delta_{\mathbf{k}}(\eta) = \frac{\eta}{\omega^2\eta^2 + 2}\left[C_1(\omega\eta\cos\omega\eta - 2\sin\omega\eta) + C_2(\omega\eta\sin\omega\eta + 2\cos\omega\eta)\right]. \tag{4.38}$$

For both cases where $C_1 \neq 0$, $C_2 = 0$ and $C_1 = 0$, $C_2 \neq 0$, we have sinusoidal behaviour in the long-term. The plot of (4.38) is given in the Figure 4.2. We see that in the radiation dominated era, we only have oscillating solutions that are neither growing nor decaying. This is consistent with our current knowledge that the universe was almost perfectly homogeneous and isotropic prior to the last scattering which occured when the universe was about 380,000 years old, where the radiation-dominated era ended when the universe was about 70,000 years old.

[3]This is of course consistent with the speed of acoustic sound wave being $1/\sqrt{3}$ of the speed of light.

Chapter 5

Conclusion

We have reviewed the work of Bičák, Katz and Lynden-Bell [4] which incorporates Mach's principle into linear cosmological perturbation theory. They have defined a Machian gauge which is a combination of the "uniform-Hubble-constant" condition of Bardeen [2] and the "minimal-distortion" condition of Smarr and York [19]. In this gauge, field equations partially decouple so that we can obtain explicit solutions for the metric perturbations. Hence, this gauge incorporates Mach's principle in the sense that the acceleration (3.14) and rotation (3.15) of local inertial frames are determined by the perturbed energy-momentum tensor $\delta T_{\mu\nu}$. However, we should always be aware if these gauge conditions are just coordinate artifacts, and if not, during which epochs they are valid. The uniform-Hubble-constant condition assumes constant mean extrinsic curvature foliations of spacetime; a global existence of such foliations is well-justified in globally hyperbolic spacetimes [20]. However, although the minimal-distortion condition is well-recognized, the other gauge condition (3.27) needs further justification.

As well as being justified mathematically and geometrically, any potential gauge must be able to reproduce the structure formation history. In §4.2, we have seen that the density perturbations oscillates and grow in the radiation- and matter-dominated era respectively, which is consistent with our current knowledge. This is, of course, a simple model using the one-fluid approximation. We may improve the model by considering a two-component model, e.g., by including cold dark matter. Nevertheless, the results we have obtained give a partial indication that Machian gauge could provide a useful foliation of hypersurfaces. Of course, one needs to also test it in various ways, e.g., by calculating the angular power spectrum and comparing with that of ΛCDM model, and checking to see if it matches with the actual structure formation history. This is left as future work.

As with any other perturbation theories, our scheme breaks down when the second order terms become significant as the inhomogeneities grow. So one always needs to bear in mind that this theory is not exact and cannot be extrapolated up to today. However, it is indeed a good approximation at the epoch of last scattering when the universe was indeed almost homogeneous and isotropic. This theory enables us to understand how primordial fluctuations lead to structure formation rather than giving the full exact history. Some gauges such as the synchronous and Newtonian gauges are successful in the sense that they are simple and consistent with many tests. However, even if they satisfy every test, we cannot resolve the problem that there does not exist a preferred foliation by hypersurfaces. Also, we should not always use them just because they make calculations simple. Machian gauges are much more restrictive than these and possibly shed light on more physical questions, other than just the implementation of Mach's principle.

All these attempts of defining a new gauge were to incorporate Mach's principle. Can Mach's principle ever be justified? We do not know, but at present, we have no experiment designed to solely test Mach's principle. Even if we had, it would probably need to be performed at some global scale, whereas we are not even sure if general relativity applies at all scales. Hence, such an experiment will never be accomplished until we have a global theory of gravity. Another issue is that, because Mach's idea was so broad, we do not have a preferred interpretation of the principle. Hence, although our primary goal would be to correctly study and understand the outcomes of the given interpretation, we must not hesitate to appreciate other possible definitions too.

We have considered Bondi's formulation: "Local inertial frames are determined through the distributions of energy and momentum in the Universe by some weighted averages of the apparent motions." One application of this is inhomogeneous cosmology, as the phrase 'by some weighted averages' in Mach's principle naturally relates to the notion of averaging in inhomogeneous cosmology. In fact, since geodesics cross during structure formation, the universe is not globally hyperbolic. The use of foliations in the late epoch universe is therefore intimately related to understanding the averaging problem and the dust approximation. Since there appears to be an average notion of homogeneity, with a preferred rest frame in which the CMB is isotropic, one slicing might be preferred over others describing the average dynamics. For this reason, the possibility of the existence of Machian gauges is an interesting issue. Wiltshire [27] has advocated for various reasons that the Machian slicing is a natural choice. There are also other attempts to implement Mach's principle. Based on his earlier work with Bertotti [28], Barbour [29] proposed another possible definition of Mach's principle by considering the field equations as a direct consequence of the Mach's principle in the case of spherical universe. Also, Gryb [30] reformulated their idea to implement Mach's principle for non-relativistic particles. Is there any physical reason why we should implement Mach's principle? Probably not. But as we have revolutionized the way of viewing the world from the one that Newton had imagined to the one that Einstein imagined, we should be open to these ideas with an ultimate goal; that is, to understand the correct nature of the universe.

Acknowledgements

I would like to thank Assoc. Prof. David L. Wiltshire, the supervisor of the project, for all his teaching and many detailed comments, and Dr Teppo Mattsson, the co-supervisor of the project, for the discussion on deciding possible applications of the project.

I would also like to thank the members of the gravity group for the general discussions, especially Dr Jenni Adams for the general advices throughout the year, and Ahsan Nazer for leading the cosmological perturbation study group at the beginning of the year which had guided me to study Weinberg's book in more detail.

Bibliography

[1] E. M. Lifshitz, J. Phys. **10** (1946) 116.

[2] J. M. Bardeen, Phys. Rev. D **22** (1980).

[3] S. W. Hawking, Astrophys. J. **145** (1966).

[4] J. Bičák, J. Katz, D. Lynden-Bell, Phys. Rev. D **76** (2007) 063501

[5] J. Barbour, H. Pfister, *Mach's Principle: From Newton's Bucket to Quantum Gravity*, (Birkhauser, Woodbine, 1995).

[6] J. Lense, H. Thirring, Phys. Z **19** (1918).

[7] F. Embacher, J. Phys. Conf. Ser. **82** (2007) 012003.

[8] W. Rindler, Phys. Lett. A **187** (1994) 235.

[9] H. Bondi, J. Samuel, Phys. Lett. A **228** (1997) 121.

[10] H. Bondi, *Cosmology*, (Cambridge University Press, Cambridge, 1961).

[11] G. F. R. Ellis, M. Bruni, Phys. Rev. D **40** (1989).

[12] T. Buchert, Gen. Rel. Grav. **32** (2000).

[13] T. Buchert, Gen. Rel. Grav. **33** (2001).

[14] D. L. Wiltshire, Phys. Rev. D **78** (2008) 084032.

[15] S. Weinberg, *Cosmology*, (Oxford University Press, Oxford, 2008).

[16] V. F. Mukhanov, H. A. Feldman, R. H. Brandenberger, Phys. Rept. **215** (1992).

[17] D. L. Wiltshire, Class. Quant. Grav **28** (2011) 164006.

[18] S. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*, (Addison–Wesley, 2004)

[19] L. Smarr, J. W. York, Phys. Rev. D **17** (1978).

[20] A. D. Rendall, Living Rev. Rel. **8** (2002) 6.

[21] R. M. Wald, *General Relativity*, (The University of Chicago Press, Chicago and London, 1984).

[22] S. Räsänen, JCAP **11** (2006) 003.

[23] M. Morita, T. Buchert, A. Hosoya, N. Li, AIP Conf. Proc. **1241** (2010) 1074-1082.

[24] E. Bertschinger, astro-ph/9503125

[25] E. Bertschinger, astro-ph/0101009.

[26] G. B. Aftken, H. J. Weber, *Mathematical Methods for Physicists, sixth edition*, (Elsevier Academic Press, Burlington, 2005)

[27] D. L. Wiltshire, New. J. Phys. **9** (2007) 377.

[28] J. B. Barbour, B. Bertotti, Proc. R. Soc. A **382** (1982).

[29] J. B. Barbour, arXiv:1108.3057 [gr-qc]

[30] S. B. Gryb, Phys. Rev. D **80** (2009) 024018.

[31] J. Traschen, D. M. Eardley, Phys. Rev. D **34** (1986) 1665.
D. J. Raine, Mon. Not. R. Astr. Soc. **171** (1975) 507-528.

Appendix A

Machian gauge condition on the vorticity

We consider another condition on the hypersurfaces differing from (3.18) but keeping (3.27). In classical fluid mechanics, the vorticity is given by $\omega = \nabla \times \mathbf{u}$ where $\mathbf{u}$ is the velocity vector. The vorticity must be transverse, i.e., $\nabla \cdot \omega = 0$, since the divergence of a curl is zero. Motivated by this, we apply this idea to our framework by demanding $\nabla_i \tilde{\omega}_{ij} = 0$. The vorticity of the general congruence of timelike world-lines that define the fluid flow is given by

$$\tilde{\omega}_{\mu\nu} = \frac{1}{2} P^{\sigma}_{\mu} P^{\rho}_{\nu} (\tilde{n}_{\sigma;\rho} - \tilde{n}_{\rho;\sigma}) \tag{A.1}$$

where $\tilde{n}_{\mu}$ is the timelike normal vector given by (3.21), which has the components:

$$\tilde{\omega}_{00} = \tilde{\omega}_{i0} = 0, \quad \tilde{\omega}_{ij} = a(\tilde{h}_{i0,j} - \tilde{h}_{j0,i}). \tag{A.2}$$

Now, demanding it to be transverse[1],

$$\nabla_i \tilde{\omega}_{ij} = a(\nabla_{ij} \tilde{h}_{i0} - \nabla^2 \tilde{h}_{j0}) = 0. \tag{A.3}$$

The above condition implies $\nabla_j \tilde{h}_{i0} - \nabla^2 \tilde{h}_{j0} = 0$. Hence, we have obtained the differential equation for $\tilde{h}_{i0}$ which we may solve analytically by using Green's functions. Now if we look at the equation (2.18),

$$8\pi G a^2 \delta T^0_i = \frac{1}{2} \nabla^2 \tilde{h}_{i0} + \frac{1}{6} \nabla_{ij} \tilde{h}_{j0} + \frac{2}{3} \nabla_i \mathcal{K}, \tag{A.4}$$

we see that we can solve for $\mathcal{K}$ since we know the left hand side and $\tilde{h}_{i0}$. Subsequently, we obtain the equation for $\tilde{h}_{nn}$ from (2.17):

$$8\pi G a^2 \delta T^0_0 = \frac{1}{3} \nabla^2 \tilde{h}_{nn} - 2\mathcal{H}\mathcal{K} \tag{A.5}$$

So we may solve for $\tilde{h}_{nn}$. With $\tilde{h}_{nn}$, $\tilde{h}_{i0}$ and $\mathcal{K}$ known, we may solve the equation

$$\mathcal{K} = \frac{3}{2} \mathcal{H} \tilde{h}_{00} + \frac{1}{2} (\tilde{h}_{nn})' - \nabla_i \tilde{h}_{i0} \tag{A.6}$$

[1] The partial derivatives are replaced by covariant derivatives, since $\tilde{\omega}_{ij}$ is antisymmetric whereas $\Gamma^i{}_{jk} = \Gamma^i{}_{(jk)}$.

for $\tilde{h}_{00}$. Finally, we can determine $\tilde{h}^T_{ij}$ by solving (2.20) and then combine with $\tilde{h}_{nn}$ to recover $\tilde{h}_{ij}$. Hence, we have determined all of the perturbed metric components directly from the perturbed energy-momentum tensor. Therefore, the gauge condition (A.3) together with (3.17) characterize another possible Machian gauge. The expressions for the explicit solutions are yet to be found, but again, one may do this by the use of Green's functions.

Since geodesic congruences orthogonal to global hypersurfaces have vanishing vorticity, the gauge conditions discussed here are not directly applicable to a perturbed FLRW geometry with a globally hyperbolic metric. However, they might be relavant for other circumstances associated with the averaging problem in the inhomogeneous universe.

Appendix B

Green's functions of the field equations

A Green's function $G(x,x')$ of a linear differential operator L is any solution of

$$LG(x,x') = \delta(x'-x), \tag{B.1}$$

where δ is the Dirac delta function. Then the solution of the differential equation

$$Lu(x) = f(x), \tag{B.2}$$

is given by

$$u(x) = \int G(x,x')f(x')dx'. \tag{B.3}$$

The Green's function for the Laplacian ∇^2 is given by

$$G(x^i, x'^i) = -\frac{1}{4\pi}\frac{1}{|x^i - x'^i|} \tag{B.4}$$

where $x^i = (x,y,z)$ are Cartesian coordinates. Then the equations (3.23) and (3.40) can be then solved given that the right hand sides are known. However, the constraint equations in $\mathbb{H}^3$ and $\mathbb{S}^3$ take more complicated form. However, such Green's functions can be found in [31].